Exact solution of a problem of dynamic deformation and nonlinear stability of a problem with a Blatz-Ko material

Edouard DIOUF

Abstract – The aim of the paper is to study the phenomena of stability of a hollow tube subjected to combined deformations. The model used is that of Blatz-Ko in compressible and dynamic. Before studying the stability, we have solved a boundary value problem with exact solution. The results could be applied in biomechanics.

Keywords –systems of nonlinear equation, exact solution in dynamic, strength theory of Blatz-Ko material, stability theory of structure

1. INTRODUCTION

For the mathematical description of many mechanical or physical problems, it is often necessary to solve equations or systems of differential equations, to search for periodic or stationary solutions to study their stability properties. The first results of the stability theory of nonlinear equations with Lyapunov appeared in the late 19^{th} century and early 20^{th} century. He then gives a sufficient condition for stability of nonlinear systems. Chataev, meanwhile, will show a theorem of instability. Massera demonstrates a necessary and sufficient condition for stability. Hahn and Lefschetz will contribute to this theory [1]

Subsequently, many theories on stability have been developed.

The first step in the qualitative theory of autonomous differential equations is an analysis of fixed points and their stability. This leads on naturally to a study of how the behavior near such fixed points can change as a parameter is varied [2].

There may be some parameters for which the system behavior changes from one qualitative state to another (the attractor of the system was a state of equilibrium and becomes a cycle for example [3,4,5].

In this work, we use non-autonomous differential equations. Initially, we are concerned with axisymmetric finite axial shear deformation of an isotropic compressible dynamic nonlinear elastic hollow circular cylinder.

Studies on the stability of such structures have been developed. For example the buckling and postbuckling of cylindrical shells under combined loading of external pressure and axial compression are demonstrated [6]. The instability analysis of a circular and thick cylinder under hydrostatic pressure is also studied [7].

The stabilization of the functionally graded cylindrical shell under axial harmonic loading is investigated by Ng and al. [8]. The authors such as Roxburgh and Ogden [9], Vandyke and Wineman [10] studied the vibrations and stability of finite deformed compressible materials.

The aim of this paper is firstly looking for an exact solution of the problem of radial deformation and axial shear and also the study of the stability of the solution.

For finding the exact solution, the purpose of the present paper is to further examine this issue for compressible materials as the solid Blatz-Ko material. We restrict the domain geometry to be that of a circular cylinder.

For a circular cylinder composed of an arbitrary isotropic incompressible elastic material, an exact solution to the axial shear problem given rise to axisymmetric anti-plane shear deformation has been obtained by Adkins [11].

Less than in the case of incompressible materials, exact solutions were obtained for compressible materials in the static case such as the finite torsion and shearing of a compressible and anisotropic tube [12] or dynamic as is the case, for example of the finite azimuthal shear motions of a transversely isotropic compressible elastic and prestressed tube [13]. To study stability, we use the technique of perturbation [14].

The question one might ask is whether we perturb a system of differential equations, the solutions obtained do they change much?

If the solutions of the perturbed system are all close to the

Edouard DIOUF, Département de Mathématiques, Université de Ziguinchor, BP 523 Ziguinchor Sénégal

solutions of the system starting, we talk about stability. When some disturbances create new solutions, instead of starting solutions, we will talk of instability. We've taken the advantage of not giving a general definition of stability for nonlinear differential equations, in order to allow the concept of stability its multifaceted character.

2. Formulation

Rubber or other polymer materials are said to be hyperelastic [15]. Usually, these kind of marterials undergo large deformations. In order to describe the geometrical transformation problems, the deformation gradient tensor is introduced by

$$\mathbf{F} = \mathbf{I} + \overrightarrow{\mathrm{grad}}\,(\mathbf{u}), \qquad (2.1)$$

where $\mathbf{I}$ is the unity tensor and $\mathbf{u}$ the displacement vector. Because of large displacements, Green-Lagrangian strain is adopted for the nonlinear relationships between strains and displacements [15]. We note the right and left Cauchy-Green deformation tensor respectively by $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$. The Green-Lagrangian strain tensor $\mathbf{E}$ is defined by

$$\mathbf{E} = (\mathbf{C} - I)/2, \qquad (2.2)$$

We are concerned with axisymmetric finite radial deformation and axial shear of an isotropic compressible nonlinear elastic hollow circular cylinder. Thus the deformation, which takes the point with cylindrical polar coordinates (R, Θ, Z) in the undeformed region to the point (r, θ, z) in the deformed region, has the form

$$r = r(R,t), \quad \theta = \Theta, \quad z = \lambda Z + h(R,t) \qquad (2.3)$$

Where $r(R,t)$ and $h(R,t)$ are unknown functions to determine and represent respectively the radial deformation and axial shear.

With respect to the cylindrical polar coordinates system, the physical components of the deformation gradient tensor is given by

$$\mathbf{F} = \begin{bmatrix} r'(R,t) & 0 & 0 \\ 0 & \dfrac{r(R,t)}{R} & 0 \\ h'(R,t & 0 & \lambda \end{bmatrix}, \qquad (2.4)$$

The Cauchy-Green tensors $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ are given by

$$\mathbf{B} = \begin{bmatrix} (r'(R,t))^2 & 0 & (r'(R,t)).(h'(R,t)) \\ 0 & (r(R,t))^2 + \dfrac{(r(R,t))^2}{R^2} & 0 \\ (r'(R,t)).(h'(R,t)) & 0 & \lambda^2 + (h'(R,t))^2 \end{bmatrix},$$
(2.5)

$$\mathbf{C} = \begin{bmatrix} (r'(R,t))^2 + (h'(R,t))^2 & 0 & \lambda h'(R,t) \\ 0 & \dfrac{(r(R,t))^2}{R^2} & 0 \\ \lambda h'(R,t) & 0 & \lambda^2 \end{bmatrix}, \qquad (2.6)$$

where $X' = \partial X / \partial R.$

In the case of hyperelastic law, there exists a strain energy density function W which is a scale function of one of the strain tensors, whose derivative with respect to a strain component determines the corresponding stress component. This be expressed by the second Piola-Kirchoff stress tensor

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} \qquad (2.7)$$

This gives in the cas of isotropic hyperelasticity [16]

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = 2\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{I} - \frac{\partial W}{\partial I_2}\mathbf{C} + I_3\frac{\partial W}{\partial I_3}\mathbf{C}^{-1}\right] \qquad (2.8)$$

where $I_1 = \mathrm{tr}(\mathbf{C}), I_2 = \left((\mathrm{tr}(\mathbf{C}))^2 - \mathrm{tr}(\mathbf{C}^2)\right)/2,\ I_3 = \sqrt{J} = \det(\mathbf{C}),$

denote the invariants of $\mathbf{C}$ and $\mathbf{B}$.

In this study, we use a Blatz-Ko material. For a solid Blatz-Ko material, undergoing a deformation characterized by a deformation gradient $\mathbf{F}$, the strain-energy function W is given by [17]

$$W=\frac{\mu}{2}\left[I_1-3+\beta\left(I_3^{1/\beta}-1\right)\right], \tag{2.9}$$

where μ and β are constants.

The blatz-Ko models for polyurethane rubber have been extensively used to describe the behavior of compressible hyperelastic isotropic material undergoing deformation [18]. By deriving the energy density (2.9) with respect to the three invariants I_1, I_2, I_3, and reporting the result in (2.8), we obtain

$$\mathbf{S}=2\mu\left[J\,\mathbf{C}^{-1}+\frac{1}{2}\mathbf{I}\right] \tag{2.10}$$

Taking into account the kinematics (2.3) and definitions of the invariants of $\mathbf{C}$, we get:

$$\begin{aligned} I_1&=\mathrm{tr}(\mathbf{B})=\lambda^2+\left(r'(R,t)\right)^2+\left(h'(R,t)\right)^2+\frac{\left(r(R,t)\right)^2}{R^2},\\ I_3&=\det(\mathbf{B})=J^2=\frac{\lambda^2\left(r(R,t)\right)^2\left(r'(R,t)\right)^2}{R^2}, \end{aligned} \tag{2.11}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ is calculated from the second Piola-Kirchoff stress tensor $\mathbf{S}$ as follows:

$$\boldsymbol{\sigma}=\frac{1}{J}\mathbf{FSF}^{\mathrm{T}} \tag{2.12}$$

which allows for taking into (2.9) and (2.10), as follows from $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}=\beta_0\mathbf{1}+\beta_1\mathbf{B}, \tag{2.13}$$

The elastic response functions $\beta_s=\beta_s(I_1,I_3)$, $(s=0,1)$ are given by [19]

$$\begin{aligned} \beta_0&=2J^{-1}I_3W_3,\\ \beta_1&=2J^{-1}W_1, \end{aligned} \tag{2.14}$$

where $W_i=\partial W/\partial I_i$, $(i=1,3)$, and we set $\beta=2$ in (2.9).

Substituting (2.5), (2.9) and (2.14) in (2.13), we obtain the components of Cauchy stress tensor

$$\begin{aligned} \sigma_{rr}&=\frac{\mu R}{\lambda r(R,t)r'(R,t)}\left[\frac{\lambda r(R,t)r'(R,t)}{R}+\left(r'(R,t)\right)^2\right],\\ \sigma_{\theta\theta}&=\frac{\mu}{\lambda R r'(R,t)}\left[\lambda^2 R r'(R,t)+\left(r'(R,t)\right)\right],\\ \sigma_{zz}&=\frac{\mu R}{\lambda r(R,t)r'(R,t)}\left[\lambda^2+\frac{\lambda r(R,t)r'(R,t)}{R}+\left(h'(R,t)\right)^2\right],\\ \sigma_{rz}&=\frac{\mu R h'(R,t)}{\lambda r(R,t)},\ \sigma_{r\theta}=\sigma_{\theta z}=0. \end{aligned} \tag{2.15}$$

The motion equations in the absence of body forces are

$$\mathrm{div}(\boldsymbol{\sigma})=\mathbf{a}\rho_0/J, \tag{2.16}$$

where $\mathbf{a}$ is the acceleration and ρ_0 the mass density in the reference state. Equations (2.16), for the deformation (2.3), reduce to the following two equations:

$$R^2 r''(R,t)+R r'(R,t)-r(R,t)-\frac{\rho_0}{\mu}R^2\frac{\partial^2 r(R,t)}{\partial t^2}=0, \tag{2.17.a}$$

$$R h''(R,t)+h'(R,t)-\frac{\rho_0}{\mu}R\frac{\partial^2 h(R,t)}{\partial t^2}=0. \tag{2.17.b}$$

To solve the system (2.17), we propose forms of $r(R,t)$ and $h(R,t)$ below [12, 13]

$$\begin{aligned} r(R,t)&=f_1(R)\cos(wt)+f_2(R),\\ h(R,t)&=h_1(R)\cos(wt)+h_2(R). \end{aligned} \tag{2.18}$$

where $f_1(R)$, $f_2(R)$, $h_1(R)$ and $h_2(R)$ are unknowns functions to determine.

Substituting equations (2.18) in those of (2.17), we obtain a system of four equations decoupled

$$f_1''(R)+\frac{1}{R}f_1'(R)+\left(m^2-\frac{1}{R^2}\right)f_1(R)=0, \tag{2.19.a}$$

$$f_2''(R)+\frac{1}{R}f_2'(R)-\frac{1}{R^2}f_2(R)=0, \qquad (2.19.b)$$

$$h_1''(R)+\frac{1}{R}h_1'(R)+m^2h_1(R)=0, \qquad (2.19.c)$$

$$h_2''(R)+\frac{1}{R}h_2'(R)=0. \qquad (2.19.d)$$

where $m^2=\rho_0 w^2/\mu$.

The boundary conditions are

$$\begin{aligned} &r(R_i,0)=R_i,\\ &r(R_o,0)=R_o,\\ &h(R_i,0)=0,\\ &h(R_o,0)=0,\\ &\sigma_{rr}(R_o,t)=0. \end{aligned} \qquad (2.20)$$

where R_i and R_o denote the inner and outer radii in the undeformed configuration.

3. Resolution

Seeking the exact solutions of nonlinear partial differential equation play an important role in the nonlinear problems [20]. This is the case of Bessel type equations encountered in many mechanical problems, particularly those with cylindrical symmetry.

Equation (2.19.a) is a Bessel equation whose solution is:

$$f_1(R)=\beta_0 J_1(mR)+\beta_1 Y_1(mR), \qquad (3.1)$$

where β_0 and β_1 are constants of integration determined from the boundary conditions (2.15), and $J_1(x), Y_1(x)$ of Bessel functions of first order, respectively first and second kind defined by

$$\begin{aligned} J_1(x)&=\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(1+k)!}\left(\frac{x}{2}\right)^{2k},\\ Y_1(x)&=\frac{2}{\pi}J_1(x)\mathrm{Log}\left(\frac{x}{2}\right)-\frac{1}{\pi}\left[\frac{2}{x}+\Psi(1)\frac{x}{2}\right]. \end{aligned} \qquad (3.2)$$

where $\Psi(n)=\dfrac{\Gamma'(n)}{\Gamma(n)}$ [21] is the logarithmic derivative of the gamma function.

The solution of the equation (2.19.b) is given by

$$f_2(R)=\overline{\beta}_0 R+\frac{\overline{\beta}_1}{R}, \qquad (3.3)$$

The solutions of equations (2.19.c) and (2.19.d) are respectively

$$h_1(R)=\overline{\overline{\beta}}_0\, J_0(mR), \qquad (3.4)$$

$$h_2(R)=-\frac{1}{2}R^2+\overline{\overline{\beta}}_1, \qquad (3.5)$$

Where $J_0(x)=\sum_{k=0}^{\infty}\dfrac{(-x^2/4)^k}{k!(1+k)!}$, and $\overline{\beta}_0, \overline{\beta}, \overline{\overline{\beta}}_0, \overline{\overline{\beta}}_1$ are constants of integration determined from the boundary conditions (2.20).

$J_1(x)$ and $J_0(x)$ are series of functions of the form $\sum_{n\geq 0} f_n(x)$, where $f_n(x)=\left(\dfrac{x}{2}\right)^{2n+1}\dfrac{(-1)^k}{k!(1+k)!}$ are defined on the same interval $I=[R_i,R_o]$. The series $\sum_{n\geq 0} f_n(x)$ converges normally on the interval I [22].

Moreover, the functions $f_n(x)$ are differentiable on I and the series of function $\sum_{n\geq 0}\left(\dfrac{\mathrm{d}f_n(x)}{\mathrm{d}x}\right)$ is also normally convergent on I. It follows that $\sum_{n\geq 0} f_n(x)$ is differentiable on I and we get $\dfrac{d}{\mathrm{d}x}\left[\sum_{n\geq 0} f_n(x)\right]=\sum_{n\geq 0}\left(\dfrac{\mathrm{d}f_n(x)}{\mathrm{d}x}\right)$ [22].

This is what gives meaning to the last equation of boundary conditions (2.20).

The Cauchy-Lipchitz theorem [23] applied to the system consisting of (2.17) (2.19) and (2.20) ensures the existence and uniqueness of solutions (3.1) (3.3) (3.4) and (3.5).

Thus, the deformation defined in (2.3) for $(R,t)\in I\times[0,+\infty[$, is given by:

$$r(R,t)=\left[\beta_0 J_1(mR)+\beta_1 Y_1(mR)\right]\cos(wt)+\overline{\beta}_0 R+\frac{\overline{\beta}_1}{R},$$
$$h(R,t)=\overline{\overline{\beta}}_0 J_0(mR)\cos(wt)-\frac{1}{2}R^2+\overline{\overline{\beta}}. \quad (3.6)$$

Using the Cauchy-Lipchitz theorem, we can define the concept of maximal solution, i.e. solutions defined on an open interval $I\times O\subset I\times[0,+\infty[$ which can not be extended to solutions over a range greater than $I\times O$.

4. Stability

This section is devoted to the concept of stability for the equations (2.17) and their solutions. This is a first step to disrupt the equations [14] to find new solutions, relatives or not starting solutions. This will serve as a basis for discussion on the stability or instability of certain solution [24, 25]. However, we took advantage of not giving a general definition of stability, to allow his character to this multifaceted concept.

To find the evolution of the fluctuations around solutions (3.6), we apply the techniques of disruptions by asking

$$\begin{aligned}\tilde{r}&=r(R,t)+\varepsilon.F_\varepsilon(R,t),\\ \tilde{z}&=\lambda.Z+h(R,t)+\varepsilon.H_\varepsilon(R,t),\end{aligned} \quad (4.1)$$

where $r(R,t), h(R,t)$ are the solutions defined en (3.6), ε is a small parameter quantifying the magnitude of disturbance and $F_\varepsilon(R,t), H_\varepsilon(R,t)$ are unknown function to determine.

We set as a new kinematics

$$\begin{aligned}\tilde{r}&=r(R,t)+\varepsilon.F_\varepsilon(R,t),\quad \tilde{\theta}=\Theta,\\ \tilde{z}&=\lambda.Z+h(R,t)+\varepsilon.H_\varepsilon(R,t).\end{aligned} \quad (4.2)$$

In this new kinematics, we will note the gradient of deformation tensor

$$\mathbf{F}_\varepsilon=\begin{bmatrix}\varepsilon\dfrac{\partial F_\varepsilon(R,t)}{\partial R}+r'(R,t) & 0 & 0\\ 0 & \dfrac{\varepsilon F_\varepsilon(R,t)+r(R,t)}{R} & 0\\ \varepsilon\dfrac{\partial H_\varepsilon(R,t)}{\partial R}+h'(R,t & 0 & \lambda\end{bmatrix},$$

the Cauchy-Green tensors $\mathbf{B}_\varepsilon=\mathbf{F}_\varepsilon\mathbf{F}_\varepsilon^{\mathrm{T}}$ and $\mathbf{C}_\varepsilon=\mathbf{F}_\varepsilon^{\mathrm{T}}\mathbf{F}_\varepsilon$,

and the Cauchy stress tensor

$\boldsymbol{\sigma}_\varepsilon=\beta_{0\varepsilon}\mathbf{1}+\beta_{1\varepsilon}\mathbf{B}_\varepsilon$, where

$$\beta_{0\varepsilon}=2J_\varepsilon^{-1}I_3W_3,\ \beta_{1\varepsilon}=2J_\varepsilon^{-1}W_1,\ J_\varepsilon=\sqrt{I_3}=\sqrt{\det(\mathbf{B}_\varepsilon)}.$$

The perturbed equation of motion is given by

$$\operatorname{div}(\boldsymbol{\sigma}_\varepsilon)=\mathbf{a}_\varepsilon\rho_0/J_\varepsilon, \quad (4.3)$$

where $\mathbf{a}_\varepsilon$ is the acceleration due to the kinematics (4.1).

By neglecting all quadratic terms (in $\varepsilon^n, n\geq 2$), in $\sigma_{ls},(l,s=r,\theta,z)$, the equations of motion (4.2) become of the form $A+\varepsilon B=0,\ \forall\varepsilon$, which gives $A=0, B=0$. The equations of motion (4.3) will then reduce to:

$$r'(R,t)\,r(R,t).\begin{bmatrix}\mu R^2 r''(R,t)+\mu Rr'(R,t)\\ -\mu r(R,t)-\rho_0 R^2\dfrac{\partial^2 r(R,t)}{\partial t^2}\end{bmatrix}=0, \quad (4.4.a)$$

$$\begin{aligned}&\mu R^2\, r(R,t)\,r'(R,t)\,F_\varepsilon''(R,t)\\ &+\begin{pmatrix}-\mu R^2 r''(R,t)+\\ \mu r(R,t)\end{pmatrix}r(R,t)\,F_\varepsilon'(R,t)+\\ &-\rho_0 R^2 r(R,t)r'(R,t)\frac{\partial^2 F_\varepsilon(R,t)}{\partial t^2}+\\ &\begin{bmatrix}-\mu r'(R,t)\left(r'(R,t)+R^2 r''(R,t)\right)+\\ \rho_0 R^2\left(r(R,t)+r'(R,t)\right)\dfrac{\partial^2 r(R,t)}{\partial t^2}\end{bmatrix}F_\varepsilon=0,\end{aligned} \quad (4.4.b)$$

$$r'(R,t)\,r(R,t)\begin{bmatrix}\mu R\,h''(R,t)+\mu\,h'(R,t)\\ -\rho_0 R\dfrac{\partial^2 h(R,t)}{\partial t^2}\end{bmatrix}=0, \quad (4.4.c)$$

$$\begin{aligned}&\begin{bmatrix}\mu R\,H_\varepsilon''(R,t)+\mu H_\varepsilon'(R,t)\\ -\rho_0 R\dfrac{\partial^2 H_\varepsilon(R,t)}{\partial t^2}\end{bmatrix}r(R,t)r'f(R,t)+\\ &-\begin{bmatrix}\mu\,h'(R,t)+\mu R\,h''(R,t)\\ -\rho_0 R\dfrac{\partial^2 h(R,t)}{\partial t^2}\end{bmatrix}.\begin{bmatrix}r(R,t)\,F_\varepsilon'(R,t)+\\ r'(R,t)F_\varepsilon(R,t)\end{bmatrix}=0.\end{aligned} \quad (4.4.d)$$

Methods for determine of periodic solutions have been significantly improved thanks largely to the work of M. Poincare. We can not be said, it seem, more general trigonometric solutions. There is no universal method to find such solutions. We must therefore resolve to restrict our field of study. Thus in this section, on the hand we want to prove the existence of a solution and also generalize the results obtained in (3.6). Therefore as an example of resolution, we propose for the functions $F_\varepsilon(R,t)$ and $H_\varepsilon(R,t)$ of the following:

$$F_\varepsilon(R,t) = F_\varepsilon^1(R)\cos(wt) + F_\varepsilon^2(R), \quad H_\varepsilon(R,t) = H_\varepsilon^1(R)\cos(wt) + H_\varepsilon^2(R). \tag{4.5}$$

where $F_\varepsilon^1(R), F_\varepsilon^2(R), H_\varepsilon^1(R)$ and $H_\varepsilon^2(R)$ are unknowns functions to determine

It follows the system of equations

$$\frac{d^2F_\varepsilon^2(R)}{dR^2} + \frac{1}{R}\frac{dF_\varepsilon^2(R)}{dR} + \left(m^2 - \frac{1}{R^2}\right)F_\varepsilon^2(R) = 0, \qquad (4.6.a)$$

$$\frac{d^2F_\varepsilon^1(R)}{dR^2} + \frac{1}{R}\frac{dF_\varepsilon^1(R)}{dR} - \frac{1}{R^2}F_\varepsilon^1(R) = 0, \qquad (4.6.b)$$

$$\frac{d^2H_\varepsilon^2(R)}{dR^2} + \frac{1}{R}\frac{dH_\varepsilon^2(R)}{dR} + m^2H_\varepsilon^2(R) = 0, \qquad (4.6.c)$$

$$\frac{d^2H_\varepsilon^1(R)}{dR^2} + \frac{1}{R}\frac{dH_\varepsilon^1(R)}{dR} = 0. \qquad (4.6.d)$$

Equations (4.6.a), (4.6.b), (4.6.c) and (4.6.d) correspond to equations (2.19.a), (2.19.b), (2.19.c) and (2.19. d), respectively. We deduce then the solutions (4.6):

$$F_\varepsilon^2(R) = f_1(R), \quad F_\varepsilon^1(R) = f_2(R), \quad H_\varepsilon^2(R) = h_1(R), \quad H_\varepsilon^1(R) = h_2(R). \tag{4.7}$$

The solution of the perturbed problem, taking account of (4.1) and (4.7) is given by

$$\begin{aligned}\tilde{r} &= r_\varepsilon(R,t) = \left(f_1(R) + \varepsilon\, f_2(R)\right)\cos(wt) + \left(f_2(R) + \varepsilon\, f_1(R)\right),\\ \tilde{z} &= z_\varepsilon(R,t) = \lambda Z + \left(h_1(R) + \varepsilon\, h_2(R)\right)\cos(wt) + \left(h_2(R) + \varepsilon\, h_1(R)\right).\end{aligned} \tag{4.8}$$

5. Discussion

We propose in this paragraph, to discuss the radial deformation. Provided that the same discussion and the same remarks can be applied to axial shear. Periodic solutions (3.6) and (4.8) are time-dependent. For time periodic solutions, there is a minimum time interval $T > 0$ (the period) after which the system returns to its original state.

The solution $r_\varepsilon(R,t)$ as defined in (4.8) is a continuous function in (R,t). The convergence of the series $J_1(mR)$, $Y_1(mR)$ and $J_O(mR)$ can show that the function $r_\varepsilon(R,t)$ is L-Lipchitz in the variable R: there exists a constant L such that:

$$\left| r_\varepsilon(R_a,t) - r_\varepsilon(R_b,t) \right| \le L\left|R_a - R_b\right|, \ \forall R_a, R_b \in [R_i, R_o], \tag{5.1}$$

The fact that $r_\varepsilon(R,t)$ is L-Lipchitz, gives a condition of stability or instability with respect to perturbation. We can find a constant L_0 as

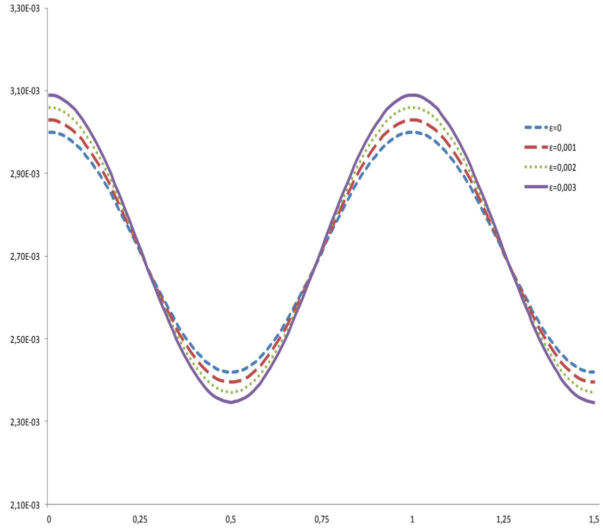
$$\left| r_\varepsilon(R,t) - r(R,t) \right| \le \varepsilon \left| f_2(R)\cos(wt) + f_1(R) \right| \le \varepsilon L_0, \ \forall R \in [R_i, R_o], \forall t \ge 0. \tag{5.2}$$

This inequality and the Cauchy-Lipchitz theorem [26, 27] show that the boundary problem (4.6) is well posed; ensure the uniqueness of the solution. In general, we can say that a solution to a problem is stable if it is insensitive to variations in data. We can interpret the sensitivity of the solution by the constant εL_0.

This sensitivity allows us to discuss the phenomena of stability. We will say that there are phenomena of instability when the constant εL_0 is high and stability when it is small. Note that when ε tends to zero, the solution (4.8) tends to the solution (3.6), i.e. to a lack of disturbance.

For the simulation, we use the numerical values of the hollow cylindrical tube: $R_i = 3mm, R_0 = 3{,}5mm$; and $w = 2\pi$ [13].

In the following figure, we present the sensitivity of the solution to perturbations.



Influence of perturbation on the radial deformation $r_\varepsilon(R_i,t)$ (m) vs. time (s)

Constant sensitivity εL_0 as defined in (5.2) can be estimated by

$$\varepsilon L_0 = \varepsilon\left[\sup_{R\in[R_i,R_o]}\left(\left|f_1(R)\right|+\left|f_2(R)\right|\right)\right]=\varepsilon(0{,}13).$$

In view of the value of εL_0, we can estimate a relatively low instability of the solution (3.6).

Note that the stability constant is proportional to disturbance. But it should be noted is that this constant is independent of time.

We answered questions about the well posedness of boundary value problems.

Note that the magnitude of R increases with the disturbance. We also note that even it is relatively small; the disturbance is a significant influence on the system.

As an application of this study, we think of a structure or prototype arterial disturbed by a disease such as an atherosclerotic plaque.

However, we noted that the stability constant is not informative to understand what happens over long times. It should also be noted that in this approach, all quadratic terms in $\varepsilon,(\varepsilon^n, n\geq 2)$ are neglected.

REFERENCES

[1] W. Hahn, *Theory and Application of liapunov's Direct Method.* Prentice-Hall inc., 1963.N.J.

[2] P.A.Glendinning, *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations*, Cambridge Texts in Applied Mathematics, CUP 1994.

[3] J.K.Hale, *Asymptotic Behavior of Dissipative Systems*, Providence: Math. Surveys and Monographs, Amer. Math.Soc.1988.

[4] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New-York 1988 (1st edition) and 1996 (2nd edition).

[5] D.Henry, *Geometric theory of Semilinear Parabolic Equation*, Springer Lecture Notes in Mathematics, Vol.840, Springer Verlag, Berlin 1984.

[6] H.S.Shen, T.Y.Chen, *Buckling and postbuckling behavior of cylindrical shells under combined pressure and axial compression*, Thin-Walled Struct. 12 (1991) 321-334.

[7] M.Barush,J.Singer, *Effet of eccentricity of stiffners on the general instability of stiffened cylindrical shells under hydrostatic pressure*, J. Mech. Eng. Sci. 5 (1963) 23-27.

[8] T.Y.Ng, Y.K.Lam, K.M.liew, J.N.Reddy, *Dynamic stability analysis of functionally grated Cylindrical shells under periodic axial loading*, Int.J. Solids Struct. 38 (2001) 1295-1300.

[9] D. G. Roxburgh and R. W. Ogden. *Stability and vibration of pre- stressed compressible elastic plates*. International Journal of Engineering Science, 32(3):427-454, 1994.

[10] T.J.Vandyke,A.S.Wineman, *Small amplitude sinusoidal disturbances superimposed on finite circular shear of a compressible, non-linearly elastic material*, Int. J. Ing. Sci.,34 (1996) 1197-1210

[11] J.E.Adkins, *Some generalizations of the shear problem for isotropic incompressible materials*, Proc. Cambridg Philos. Soc. 50, 334-345 (1954).

[12] M.Zidi,*Finite torsion and shearing of a compressible and anisotropic tube*, Int. Journal of Non linear Mechanics, 35:1115-1126 (2000).

[13] E.Diouf,M.Zidi, *Finite azimuthal shear motions of a transversely isotropic compressible elastic And pretressed tube*, Int. J. Ing. Sci., 43 :262-274 (2005).

[14] Jouve F.,*Modélisation de l'oeil en élasticité non linèaire*, Masson, Paris, 1993.

[15] Zhi-Qiang Feng, B. Magnain, J. Cros, *Solution of large deformation impact problems with friction between Blatz-Ko hyperelastic bodies.* Int. J. Ing. Sci. 44 (2006) 113-126.

[16] P.G. Ciarlet,*Elasticité tridimensionnelle,* Masson, Collection, RMA, 1985.

[17] M. Destrade,G.Saccomandi,*On finite amplitude elastic waves propagating in compressible solids.*(2005) Physical Review E.

[18] M.Destrade,*Finite-Amplitude inhomogeneous plane waves in a deformed Blatz-Ko Materal,* CanCNSM, Victoria, June 16-20, 1999.

[19] A.D. Polignone, C.O. Horgan, *Axisymmetric finite anti-plane shear of compressible nonlinealy elastic circular tubes.*Quarterly of applied mathematics, Vol.L, N.2, june 1992, Pages 323-341.

[20] C.Truesdell, W.Noll, *The non-linear field theories of mechanics,* Handbuch der Physik, III/3 (S.Flugge, ed.), Springer-Verlag, Berlin, 1965.

[21] F.Laroche,*Promenade Mathématiques, Fonctions de Bessel*, promenadesmaths.free.fr, 2004. ridge University Press, 1998.

[22] C. Deschamps, A. Warusfel,*Mathématiques Tout-en-un. 2ième année MP,* 2ième edition Dunod, 2004.

[23] A.M. Stuart, A.R.Humphries,*Dynamical systems and numerical analysis,*Cambridge University Press, 1998.

[24] E.Hebey,*Nonlinear elliptic equations of critical Sobolev growth from a dynamical viewpoint* Noncompact problems at the inter-section of geometry, analysis, and topology, Contemp. Math., vol. 350, Amer. Math.Soc., Providence, RI, 2004,p.115-125.

[25] O.Druet, E.Hebey, J. Vétois, *Boundary stability for strongly coupled critical elliptic systems below the geometyric threshold of the conformal Laplacien,*J. Funct. Anal. 258 (2010), No.3,p. 999-1059.

[26] T.Kato, *Pertubation theory for linear operator,*Classic in Mathematics, Springer-Verlag,Berlin, 1995, Reprint of the 1980 edition.

[27] A.M.S., A.R.Humphries, *Dynamical systems and numerical analysis.* Camb